

# Optimal data collection for improved rankings expose well-connected graphs

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## Abstract

Given a graph where vertices represent alternatives and arcs represent pairwise comparison data, the statistical ranking problem is to find a potential function, defined on the vertices, such that the gradient of the potential function agrees with the pairwise comparisons. Our goal in this paper is to develop a method for collecting data for which the least squares estimator for the ranking problem has maximal information. Our approach, based on experimental design, is to view data collection as a bi-level optimization problem where the inner problem is the ranking problem and the outer problem is to identify data which maximizes the informativeness of the ranking. Under certain assumptions, the data collection problem decouples, reducing to a problem of finding graphs with large algebraic connectivity. This reduction of the data collection problem to graph-theoretic questions is one of the primary contributions of this work. As an application, we study the 2011-12 NCAA football schedule and propose schedules with the same number of games which are significantly more informative. Using spectral clustering methods to identify highly-connected communities within the division, we argue that the NCAA could improve its notoriously poor rankings by simply scheduling more out-of-conference games.

**Keywords:** Bayesian experimental design, graph synthesis, algebraic connectivity, statistical ranking, scheduling.

## 1. Introduction

The problem of statistical ranking<sup>1</sup> arises in a variety of applications (and in particular competitive sports), where a collection of alternatives (teams) is to be ranked based on pairwise comparisons (games). Methods for ranking must address a number of inherent difficulties including (1) incomplete data (not all teams play all other teams), (2) inconsistencies in the data (team A beats team B, team B beats team C, and team C beats team A), and (3) the data is imbalanced (the “strength of schedule” varies amongst the teams). Despite and possibly as a consequence of these difficulties, although ranking from pairwise comparison data is an old problem [David \(1963\)](#), there have been several recent contributions to the subject [Langville and Meyer \(2012\)](#); [Osting et al. \(2012\)](#); [Hirani et al. \(2011\)](#); [Jiang et al.](#)

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1. To prevent confusion, we note that we use the term ranking to indicate a numerical score for each item in a collection, which is sometimes referred to as a rating.

(2010); Callaghan et al. (2007). In this work, we adopt the language associated with sports rankings (for example, collecting pairwise data will be referred to as scheduling games), but our results have broader applicability in data collection, *e.g.*, problems in social networking, game theory, network security, and logistics.

Our central goal in this paper is to investigate the dependence of the ranking problem on the schedule of play, which we denote by  $w$ . When viewed as a statistical inverse problem, the ranking problem is to estimate the overdetermined parameter which describes each team's strength (ranking),  $\phi$ , given (noisy) observations,  $y$ , which represents the margin of victory for all scheduled games. Symbolically, an estimator for the ranking problem is expressed

$$\hat{\phi}_w = \mathcal{R}(y, w), \quad (1)$$

where the dependence of the ranking,  $\hat{\phi}_w$ , on the schedule,  $w$ , is emphasized by the subscript.

Generally speaking, the more games played amongst a fixed number of teams (*i.e.*, the more pairwise data gathered), the more informative we expect the ranking,  $\hat{\phi}_w$ . That is, there is a tradeoff between the number of games played and the value of the ranking. For example, in a single elimination tournament with  $n$  teams, there are only  $n-1$  games played. Here, we expect that the “best team” wins the tournament, but it is difficult to rank the remaining teams in any reasonable way. At the other extreme, a round-robin tournament amongst  $n$  teams requires  $\binom{n}{2}$  games which may not be possible if  $n$  is large. In this paper, we consider the following question: For  $n$  teams playing  $m$  games, with  $n-1 < m < \binom{n}{2}$ , how can the schedule be arranged to produce the “most informative” ranking?

We follow the methodology of the optimal design community Haber et al. (2008); Pukelsheim (2006); Melas (2006); Fedorov (1972), and consider the *Fisher information* for the ranking estimate  $\mathcal{R}$ , defined in (1). Assuming  $\mathcal{R}$  is unbiased, *i.e.*,  $\mathbb{E}\hat{\phi}_w = \phi$ , the Fisher information is the inverse of the variance,  $\text{Var}(\hat{\phi}_w)$ , and thus maximizing the informativeness of the ranking is equivalent to minimizing  $\text{Var}(\hat{\phi}_w)$ . We are thus led to the following optimization problem:

$$\min_w f(\text{Var}(\hat{\phi}_w)) \quad (2a)$$

$$\text{such that } \hat{\phi}_w = \mathcal{R}(y, w) \quad (2b)$$

$$w \in \mathbb{Z}_+^N, \quad \|w\|_1 = m \quad (2c)$$

where  $N := \binom{n}{2} = \frac{n(n-1)}{2}$  and  $f: \mathbb{S}_+^n \rightarrow \mathbb{R}$  is a convex function. For general optimal design problems, common choices for the scalar function  $f(A)$  include the following:

$$f(A) = \|A\|_2 = \max_i \lambda_i(A) \quad \text{E-optimal} \quad (3a)$$

$$f(A) = \text{tr} A = \sum_i \lambda_i(A) \quad \text{A-optimal} \quad (3b)$$

$$f(A) = \det A = \prod_i \lambda_i(A) \quad \text{D-optimal} \quad (3c)$$

where  $\{\lambda_i(A)\}_{i=1}^n$  denote the eigenvalues of  $A$ . The constraint in (2c) specifies that the schedule consists of  $m$  games.

A schedule can be represented as a graph,  $G = (V, E)$ , with  $n$  nodes, denoted  $V$ , representing teams and  $m$  edges, denoted  $E$ , representing games. The schedule,  $w$ , is then an integer valued function on the edges with components defining the number of games played between the two incident teams. In §4, we show that for the least squares estimator, the constraint (2b) in the optimization problem (2) decouples, yielding a graph synthesis problem of finding the graph whose graph Laplacian has desired spectral properties. For example, an E-optimal schedule corresponds to a graph with maximal second Laplacian eigenvalue.

**Current practice** There are large variations in the methods currently used for sports scheduling. Here, we describe the type of scheduling which is the focus of this work, beginning with the following distinction:

- **static scheduling:** The schedule is determined prior to the season, independent of the performance of teams throughout the season. Examples of leagues employing static schedules include NCAA football and Major League Baseball (MLB).
- **dynamic scheduling:** The schedule dynamically changes based on score results. For example, in a single elimination tournament, a team advances to the next round only if they win in the current round. Leagues which partially rely on single elimination tournaments include ATP tennis and FIFA World Cup soccer.

While dynamic schedules incorporate the results of previous games and are thus more informative than static schedules, they have the disadvantage that they may not be completely determined prior to the season. In this paper, we focus only on static scheduling.

For statically scheduled games, the most important quantity is the ratio of the total number of games played to the total number of teams. In MLB, there are 30 teams, divided into two leagues: the American League (14 teams) and the National League (16 teams). During the regular season, each team plays approximately 160 games, primarily against teams within the same division. Thus within each league teams play an average of  $160/15 \approx 10$  times. With so many games and equal strength of schedule amongst teams, it is intuitive that the scheduling has little effect on the rankings. And, in fact, MLB simply uses win/loss percentages for ranking purposes. In NCAA football however, there are 120 teams in the NCAA Football Bowl Subdivision (FBS) and each team plays approximately 6 games per year within FBS. Thus each team only plays roughly 5% of the other teams. There are several rankings for NCAA football which are generated either mathematically or by expert opinion and then aggregated to determine official rankings and select teams to compete in the prestigious end-of-season “bowl games”. The fact that these rankings generally disagree and that none of them is more reliable than the others suggest that none of them are very informative. It is this situation, where there are relatively few games compared to the number of teams, that the schedule indeed has a large effect on the rankings. In this paper, we construct schedules for which the associated rankings are significantly more informative than the NCAA football schedule.

**Outline** In §2, we review related work. In §3, we review properties of the eigenvalues of the graph Laplacian and establish notation used in subsequent sections. In §4 we study the schedule design problem (2) and show the reduction of (2) to a graph synthesis problem. In §5, we conduct a number of numerical experiments to demonstrate properties of

nearly-optimal and randomly generated schedules. These are compared with the 2011-12 NCAA Division 1 football schedule. Finally, we conclude in §6 with a discussion of further directions.

## 2. Related work

Our work is related to three subject areas, which we discuss in turn: statistics and experimental design, sports scheduling, and graph theory.

**Statistics and experimental design** Excellent surveys of the optimal experiment design literature can be found in [Haber et al. \(2008\)](#); [Pukelsheim \(2006\)](#); [Melas \(2006\)](#); [Fedorov \(1972\)](#).

Methods of optimal experiment design have been applied to ill-posed inverse problems, *e.g.*, in geophysical [Haber et al. \(2008\)](#) or biomedical imaging ([Horesh et al., 2011](#), ch. 13, p. 273-290), [Chung and Haber \(2012\)](#); [Quinn and Keough \(2002\)](#); [DiStefano 3rd \(1976\)](#). It is instructive to consider the analogy between these applications and the scheduling design problem considered here. In imaging systems, there is a tradeoff between the amount of collected data and the accuracy of the reconstruction, or equivalently, the sparsity of the measurement and the uncertainty in the solution to the inverse problem. For application dependent reasons (*e.g.*, high radiation dose to a patient or the cost of collecting data), it is often desirable to place as few sensors as possible while still maintaining an acceptable accuracy in the reconstruction. In sports scheduling, the goal is to construct the best ranking possible from a small number of games. In both situations, it is desirable to take “measurements” which are maximally informative.

### Sports scheduling: single-elimination tournaments and active learning methods

Previous work in sports scheduling can be roughly divided into the following two categories.

The first type of scheduling focuses on the seeding policy of single-elimination tournaments with the objective of arranging the teams so that the outcome of the tournament agrees with a preexisting ranking [D’Souza \(2010\)](#); [Scarf and Yusof \(2011\)](#); [Glickman \(2008\)](#) or an arrangement which favors a particular team [Vu et al. \(2001\)](#). These objectives depend on a preexisting ranking of the teams, which we do not assume to know in this paper. Another type of tournament scheme is investigated in [Ben-Naim and Hengartner \(2007\)](#), where a sequence of rounds of diminishing size are used to determine the best team.

The second type of scheduling is a dynamic scheduling method. Games are scheduled which maximize the expected gain in information and thus one can view the resulting schedules as a greedy algorithm to learning as much as possible about the rankings [Glickman \(2005\)](#). This is the active learning approach, where past observations are used to control the process of gathering future observations, see, for example, [Krause et al. \(2008\)](#); [Seeger and Nickisch \(2011\)](#); [Silva and Carin \(2012\)](#). The current work shares the same objective with this second type of scheduling. The difference is that we do not update the expected game outcomes as the season progresses. In this case, the schedule can be fixed before the season begins. This simplification is significant and leads to a formulation of the scheduling problem which has an interesting graph theoretical interpretation. An extension of this work is the interpretation of the dynamic scheduling method as a perturbation to the static problem.

Finally, we remark that considerations of the schedule cost lead to variations of the traveling salesman problem.

**Graph theory** In this paper, we reduce the schedule design problem (2) to a graph synthesis problem. We focus on the optimality condition given in (3a), which reduces to finding graphs with maximal algebraic connectivity. There is a tremendous amount of work on the algebraic connectivity of graphs, originating with studies by Miroslav Fiedler Fiedler (1973). Many properties of algebraic connectivity are reviewed in Mohar (1991); Biyikoglu et al. (2007) and we also review some of these results in §3. The problems arising from the other optimality conditions, (3b) and (3c), are less well studied Grimmer (2010); Ghosh and Boyd (2006a); Ghosh et al. (2008).

The robustness of a network to node/edge failures is highly dependent on the algebraic connectivity of the graph. Also, the rate of convergence of a Markov process on a graph to the uniform distribution is determined by the algebraic connectivity Sun et al. (2004). Finally, in the “chip-firing game” of Björner, Lovász and Shor, the algebraic connectivity dictates the length of a terminating game Björner et al. (1991). Consequently, algebraic connectivity is a measure of performance for the convergence rate in sensor networks, data fusion, load balancing, and consensus problems Olfati-Saber et al. (2007).

### 3. Eigenvalues of the graph Laplacian and algebraic connectivity

In this section, we briefly survey relevant results on the eigenvalues of the graph Laplacian and algebraic connectivity. More extensive treatments are given in Fiedler (1973); Biyikoglu et al. (2007); Mohar (1991); Chung (1997). In §3.1, we recall algorithms for computing graphs with large algebraic connectivity Ghosh and Boyd (2006b).

Let  $B \in \mathbb{R}^{N \times n}$  where  $N := \binom{n}{2}$  be the arc-vertex incidence matrix for the complete directed graph on  $n$  nodes,

$$B_{k,j} = \begin{cases} 1 & j = \text{head}(k) \\ -1 & j = \text{tail}(k) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

where the arc orientations (heads and tails of arcs) have been chosen arbitrarily. Let  $G = (V, E)$  be a graph with  $|V| = n$ . It is convenient to represent the edge set,  $E$ , by a vector  $w \in \{0, 1\}^N$ , such that the  $k$ -th edge is present if  $w_k = 1$  and  $w_k = 0$  otherwise. We refer to  $w$  as the indicator function of the edge set. The graph Laplacian of  $G$  is defined

$$\Delta_w := B^t W B \quad \text{where } W = \text{diag}(w).$$

Let  $\lambda_i(w)$  for  $i = 1, \dots, n$  denote the eigenvalues of  $\Delta_w$ . The eigenvalues are contained in the interval  $[0, \min(n, 2d_+)]$  where  $d_+$  is the maximum degree of  $G$ . The first eigenvalue of  $\Delta_w$ ,  $\lambda_1$ , is zero with corresponding eigenvector  $v_1 = 1$ . The second eigenvalue,  $\lambda_2$ , is nonzero if and only if the graph is connected. The second eigenvalue is referred to as the *algebraic connectivity* of  $G$  and characterized by

$$\lambda_2(w) = \min_{\substack{\|v\|=1 \\ \langle v, 1 \rangle = 0}} \|Bv\|_{2,w}. \quad (5)$$

The eigenvector  $v_2$  corresponding to  $\lambda_2$  is sometimes called the Fiedler vector after Miroslav Fiedler for his contribution to the subject [Fiedler \(1973\)](#).

Let  $w_i \in \{0, 1\}^N$  for  $i = 1, 2$  be the indicator function for two edge sets  $E_i$  defined on a vertex set  $V$ . It follows from [\(5\)](#) that  $w_1 \leq w_2$  implies  $\lambda_2(w_1) \leq \lambda_2(w_2)$ . That is, for fixed vertex set,  $E_1 \subseteq E_2$  implies the more connected graph has greater algebraic connectivity.

Let  $U \subset V$  and  $\text{cut}(U, U^c)$  be the set of edges connecting  $U$  and  $U^c := V \setminus U$ . Then the algebraic connectivity is bounded by the normalized graph cut,

$$\lambda_2(w) \leq \min_{U \subseteq V} \frac{n|\text{cut}(U, U^c)|}{|U||U^c|}. \quad (6)$$

In particular, if  $U = \{v\}$  where  $v \in V$  is the node with smallest degree, i.e.,  $d_v = d_-$ , then  $d_v \leq \frac{2m}{n}$  where  $m = |E|$  and we obtain

$$\lambda_2(w) \leq \frac{nd_-}{n-1} \leq \frac{2m}{n-1}. \quad (7)$$

Properties of graphs for which the bound in [\(7\)](#) is tight have been studied [Fallat et al. \(2003\)](#). For an incomplete graph, the algebraic connectivity is bounded above by both the vertex connectivity,  $C_v(G)$ , and edge connectivity,  $C_e(G)$ ,

$$0 \leq \lambda_2 \leq C_v \leq C_e \leq d_-,$$

where  $d_-$  is the minimal vertex degree [Fiedler \(1973\)](#). The algebraic connectivity can also be bounded in terms of Cheeger's inequality, Buser's inequality, and the diameter of the graph [Biyikoglu et al. \(2007\)](#); [Mohar \(1991\)](#); [Chung \(1997\)](#).

### 3.1 Finding graphs with large algebraic connectivity

In several applications, it is useful to compute graphs with large algebraic connectivity, [\(5\)](#). The problem of finding weights  $w \in \mathbb{R}^N$  which maximize  $\lambda_2(w)$  is a convex optimization problem and can be formulated as a semidefinite program (SDP) [Ghosh and Boyd \(2006b\)](#). However, if  $w \in \mathbb{Z}_+^N$ , the problem is NP-hard [Mosk-Aoyama \(2008\)](#). This is the case arising in the schedule optimization problem.

The integer constrained problem may be solved by relaxing to the unconstrained problem and then rounding the solution. This is clearly an lower bound on the optimal solution and, if the values  $w$  are large, a reasonable approximation. Another approach, advocated by [Ghosh and Boyd \(2006b\)](#); [Wang and Mieghem \(2008\)](#), is to use the greedy algorithm based on the Fiedler vector described in [Algorithm 1](#). This algorithm adds a specified number of edges to an input graph to maximize the algebraic connectivity of the resulting augmented graph. In this work, we refer to graphs produced via this method as *nearly-optimal*.

## 4. Optimal scheduling using a least squares ranking

We assume that each team  $j = 1, \dots, n$  has a ranking (measure of strength) given by  $\phi_j$ . We label each possible game by  $k = 1, \dots, \binom{n}{2} \equiv N$  and denote by  $B \in \mathbb{R}^{N \times n}$  the arc-vertex incidence matrix [\(4\)](#) for the complete graph. For each pair of teams  $k = \{i, j\}$ , we assume

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**Algorithm 1:** A greedy heuristic for computing graphs with large algebraic connectivity [Ghosh and Boyd \(2006b\)](#); [Wang and Mieghem \(2008\)](#). See §3.1.

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**Data:** Given an initial graph  $G_i = (V, E_i)$ , and an integer number,  $M$ , of edges to add.

**Output:** A graph  $G = (V, E)$  with large algebraic connectivity with edge set of size  $|E| = |E_i| + M$ .

Set  $E = E_i$  (current edge set).

**for**  $k = 1$  **to**  $M$ , **do**

    Compute the Fiedler vector

$$F = \arg \min_{\substack{\|v\|=1 \\ \langle v, 1 \rangle = 0}} \|Bv\|_{2,w}.$$

    Find the edge  $\{i, j\} \notin E$  which maximizes  $(F_i - F_j)^2$ .

    Set  $E = E \cup \{i, j\}$ .

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that there exists some measure of the outcome of the games played between teams  $i$  and  $j$ , denoted  $y_k$ , such that

$$y_k = (B\phi)_k + \epsilon_k \quad (8)$$

where  $\epsilon_k$  is a random variable with zero mean, *i.e.*,  $\mathbb{E}\epsilon = 0$ . Let  $w_k \in \mathbb{Z}_+$  denote the number of games played between teams  $i$  and  $j$ . We assume that the variance of  $\epsilon_k$  is given by  $\sigma^2/w_k$  for some constant  $\sigma$ . More games between teams  $i$  and  $j$ , reduce the variance in the observed pairwise comparison. We have in mind that  $y_k$  is a function of the score differences (margin of victory) for the games played between teams  $i$  and  $j$ , but, surprisingly, our approach does not require this to be specified precisely. We *only* require the existence of such a measure of the game outcomes.

**Ranking** There are several choices for the ranking in (1). The Gauss-Markov theorem states that the least squares estimator,

$$\hat{\phi}_w = \mathcal{R}(y, w) \quad (9a)$$

$$= \arg \min_{\langle \phi, 1 \rangle = 0} \|B\phi - y\|_{2,w} \quad (9b)$$

$$= (B^t W B)^\dagger B^t W y, \quad (9c)$$

is the linear, unbiased ( $\mathbb{E}[\hat{\phi}_w] = \phi$ ) estimator with smallest variance. The variance of  $\hat{\phi}_w$  is given by

$$\text{Var}(\hat{\phi}_w) = (B^t W B)^\dagger \equiv \Delta_w^\dagger, \quad (10)$$

which is the Moore-Penrose pseudoinverse of the  $w$ -weighted graph Laplacian. Equation (10) is shown, using the linearity of the estimator, as follows. We first compute

$$\hat{\phi}_w = (B^t W B)^\dagger B^t W y = (B^t W B)^\dagger B^t W (B\phi + \epsilon) = \phi + (B^t W B)^\dagger B^t W \epsilon.$$

Thus,

$$\text{Var}(\hat{\phi}_w) = \mathbb{E}[(\hat{\phi}_w - \phi)(\hat{\phi}_w - \phi)^t] = (B^t W B)^\dagger B^t W \mathbb{E}[\epsilon \epsilon^t] W B (B^t W B)^\dagger.$$



Assuming that  $\mathbb{E}[\epsilon\epsilon^t] = \sigma^2 W^{-1}$ , we obtain (10).

The use of the least-squares estimate (9) in ranking has been referred to as HodgeRank Jiang et al. (2010) and is related to the Massey and Colley methods used in sports rankings Langville and Meyer (2012) and is the ranking method considered in the present work.

**Schedule Design** The schedule design problem (2) is to find  $w$  such that the variance of the estimate  $\hat{\phi}_w$  is minimal in the sense of the semi-definite ordering (*i.e.*,  $A \geq B$  if  $A - B \succeq 0$ ). It is remarkable that  $\text{Var}(\hat{\phi}_w)$ , given in (10) doesn't depend on the scores,  $y$ . Thus, the constraint in the optimal scheduling problem (2b) decouples. Traditional optimality criteria are functions of the eigenvalues of  $\text{Var}(\hat{\phi}_w)$  such as given in (3) Haber et al. (2008); Pukelsheim (2006); Melas (2006); Fedorov (1972). In what follows, we use the "E-optimality condition" of minimizing  $\lambda_{\max}(\text{Var}(\hat{\phi}_w))$ . Using (10), this is equivalent to maximizing the smallest non-zero eigenvalue of the  $w$ -weighted graph Laplacian,  $\Delta_w \equiv B^t W B$ . For a connected graph, the smallest non-zero eigenvalue is the second one,  $\lambda_2(w)$  defined in (5). Thus the optimal schedule is obtained by solving the following eigenvalue optimization problem

$$\begin{aligned} \max_w \quad & \lambda_2(w) \\ \text{such that} \quad & w \in \mathbb{Z}_+^N, \quad \|w\|_1 = m. \end{aligned} \tag{11}$$

For simplicity, in §5 we will further assume that each pair of teams plays at most once, *i.e.*,  $w \in \{0, 1\}^N$ . In this case,  $\lambda_2(w)$  is the *algebraic connectivity* of the graph (see §3), and (11) can be interpreted as the graph synthesis problem of finding a graph with  $n$  nodes and  $m$  edges with largest algebraic connectivity. The more general case corresponding to teams playing one another multiple times, *i.e.*,  $w \in \mathbb{Z}_+^N$ , can be interpreted as the algebraic connectivity of a multigraph. This direction is not further pursued here.

We summarize the preceding discussion with the following proposition:

**Proposition 4.1** *Let  $\epsilon$  be a random vector with  $\mathbb{E}\epsilon = 0$  and  $\text{Var}(\epsilon) = \sigma^2 W^{-1}$  where  $W = \text{diag}(w)$  and  $w \in \mathbb{Z}_+^N$ . Let  $\hat{\phi}_w$  be the least squares estimator given in (9) for  $\phi$  in (8). The schedule  $w \in \mathbb{Z}_+^N$  with  $\|w\|_1 = m$  which minimizes  $\|\text{Var}(\hat{\phi}_w)\|_2$  is the solution of the graph synthesis problem (11).*

**Remark 1** *Other measures of optimality, such as those given in (3), may be used in place of the objective function in (11). The  $D$ -optimal objective function can be interpreted as the number of spanning trees within the graph Ghosh and Boyd (2006a). The  $A$ -optimal objective function is the total effective resistance of a electric circuit constructed by identifying each edge of the graph with a resistor of equal resistance Ghosh and Boyd (2006a); Ghosh et al. (2008) and is related to the return time for a reversible Markov chain Grimmer (2010).*

**Remark 2** *The Hodge decomposition implies that the residual in (9b),  $r = B\phi - y$ , can be further decomposed into two orthogonal components: (1) a divergence-free component which consists of 3-cycles and (2) a harmonic component which consists of longer cycles Jiang et al. (2010); Hirani et al. (2011). In fact, Jiang et al. (2010) argues that a dataset which has a large harmonic component is inherently inconsistent and does not have a reasonable ranking. The harmonic component lies in the kernel of the graph Helmholtzian which has dimension given by the first Betti number of the associated simplicial complex.*




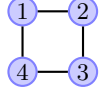
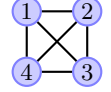
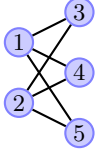
	path, $P_n$	cycle, $C_n$	complete, $K_n$	complete bipartite, $K_{n,\ell}$
diagram				
eigenvalues	$2 - 2 \cos(\pi k/n)$ $k = 0, \dots, n-1$	$2 - 2 \cos(2\pi k/n)$ $k = 0, \dots, n-1$	$0_1, n_{n-1}$	$0_1, n_{\ell-1},$ $\ell_{n-1}, (\ell+n)_1$
alg. conn. (5)	$2 - 2 \cos(\pi/n)$	$2 - 2 \cos(2\pi/n)$	$n$	$\min(n, \ell)$
edge conn.	1	2	$n-1$	$\min(n, \ell)$
diameter	$n$	$\lfloor n/2 \rfloor$	1	2

Table 1: A comparison of several measures of connectivity for 4 well-known graphs. We assume  $n \geq 3$ . Subscripts on the eigenvalues denote multiplicity and  $\lfloor \cdot \rfloor$  indicates the floor function. See §5.1.

## 5. Numerical experiments

In this section we study graphs corresponding to schedules which are good for ranking, and in particular, graphs with large algebraic connectivity. In §5.1, we consider structured graphs for which the eigenvalues of the Laplacian can be analytically computed and small graphs with  $\leq 5$  edges. In §5.2, we compare the expected algebraic connectivity of Erdős-Rényi random graphs with graphs obtained using the greedy algorithm described in §3.1. In §5.3, we discuss the algebraic connectivity for the graph corresponding to the 2011-12 NCAA Division I football schedule.

### 5.1 Algebraic connectivity for example graphs

In this section, we give results on the algebraic connectivity for graphs with easily computable spectra and graphs with a small number of nodes. In Table 1, we tabulate the eigenvalues, algebraic connectivity (5), edge connectivity, vertex connectivity, and diameter for 4 well-known graphs. The number of distinct  $n$ -node, connected, unlabeled graphs for  $n=1, 2, 3, \dots$  are 1, 1, 2, 6, 21, 112, 853, 11117, 261080,  $\dots$  (Sloane A001349). In Fig. 1 we plot, for  $n=4$  and  $n=5$ , each of these graphs together with the algebraic connectivity,  $\lambda_2$ .

In Fig. 1, we observe that as the number of edges,  $m$ , is increased, the algebraic connectivity,  $\lambda_2$ , generally increases. Furthermore, for a fixed number of edges,  $m$ , the algebraic connectivity can vary significantly. For  $m=5, 6$ , and  $7$ , the value of  $\lambda_2$  varies by a factor  $\geq 2$ . For  $m=5$ , the graph with smallest  $\lambda_2$  has small edge connectivity (and hence small algebraic connectivity) and the graph with largest  $\lambda_2$  has nodes with equal degree. These small graphs beautifully illustrate the bounds given in §3.

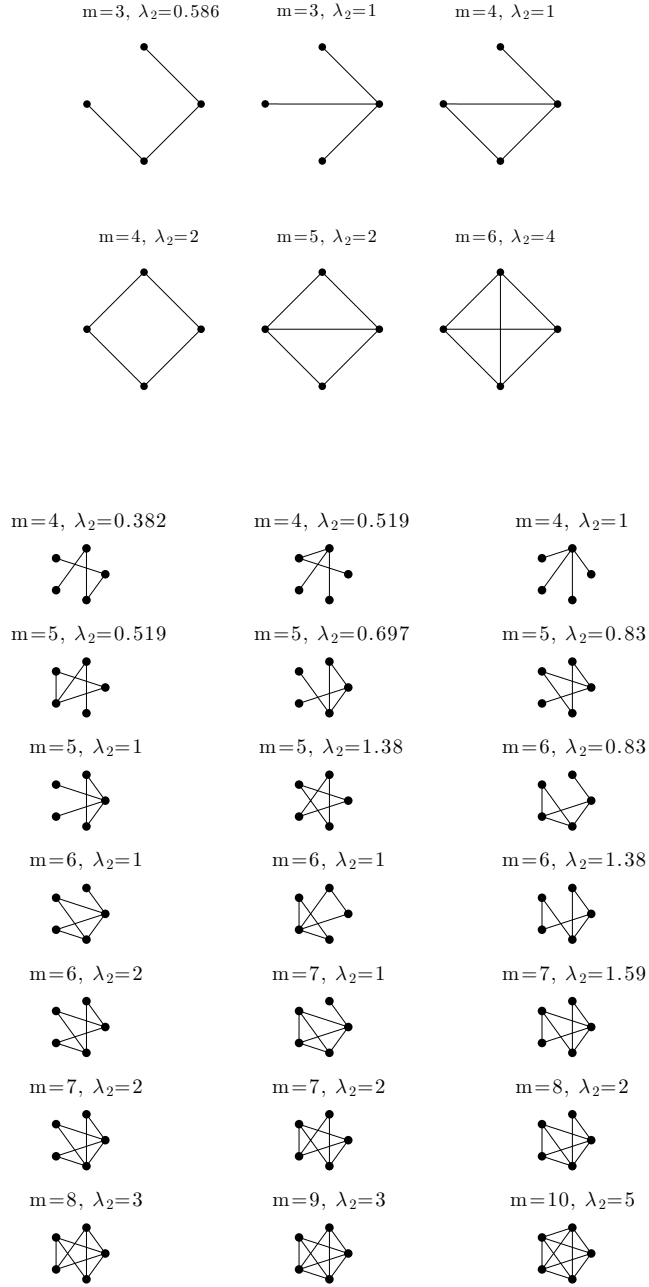


Figure 1: The 4- and 5-node connected graphs and their algebraic connectivity,  $\lambda_2$ . Graphs corresponding to more informative schedules have large algebraic connectivity. See §5.1.

## 5.2 Algebraic connectivity of Erdős-Rényi random graphs and computed nearly-optimal graphs

We consider the Erdős-Rényi random graph model  $G(n, p)$  containing graphs with  $n$  nodes and edges included with probability  $p$ , independent from every other edge. The expected number of edges for a graph in  $G(n, p)$  is  $p\binom{n}{2}$  and the threshold for connectedness is  $p_c = \frac{\log n}{n}$ .

There are several results on the spectrum of the graph Laplacian for Erdős-Rényi graphs, especially in the limit  $n \uparrow \infty$ , see [Chung and Radcliffe \(2011\)](#); [Oliveira \(2009\)](#). Finite sized random graphs are less well-understood. The algebraic connectivity of Erdős-Rényi, Watts-Strogatz, and Barabás-Albert random graphs has been studied numerically in [Jamakovic and Uhlig \(2007\)](#). The algebraic connectivity of a Watts-Strogatz graph is known to have a phase transition [Olfati-Saber et al. \(2007\)](#).

We will utilize the following elementary upper bound on the algebraic connectivity, analogous to (7), derived using a concentration inequality.

**Proposition 5.1** *Let  $\epsilon > 0$  and assume  $n$  to be even. With probability at least  $1 - \epsilon$ , the algebraic connectivity,  $\lambda_2$ , of an Erdős-Rényi graph  $G(n, p)$  satisfies*

$$\lambda_2 \leq np + 4n^{-2}\sqrt{2\log(1/\epsilon)}. \quad (12)$$

**Proof** Choose any subset  $U \subset V$  with  $|U| = \frac{n}{2}$ . Equation (6) implies that  $\lambda_2 \leq \frac{4C}{n}$  where  $C \sim \mathcal{B}(\frac{n^2}{4}, p)$ . For  $a > 0$ , we compute

$$\Pr(\lambda_2 \geq np + a) \leq \Pr(4C/n \geq np + a) = \Pr(C - pn^2/4 \geq +an/4) \leq \exp(-a^2n^4/32)$$

where the last inequality is due to Hoeffding. Setting  $a = 4n^{-2}\sqrt{2\log(1/\epsilon)}$ , we find that  $\Pr(\lambda_2 \geq np + a) \leq \epsilon$  as desired.  $\blacksquare$

**Remark 3** *For odd  $n$ , Prop. 5.1 holds, except  $n(n^2 - 1)^{-\frac{3}{2}}$  replaces  $4n^{-2}$  in (12).*

For a random graph  $G(n, p)$ , the number of edges  $m \sim \mathcal{B}(N, p)$  where  $N := n(n - 1)/2$ . Thus,  $\mathbb{E}[m] = pN$  and we may restate (12): as: with probability at least  $1 - \epsilon$ ,

$$\lambda_2 \leq \frac{2\mathbb{E}[m]}{n - 1} + 4n^{-2}\sqrt{2\log(1/\epsilon)}. \quad (13)$$

Indeed, the first term on the right hand side of (13) matches the right hand side of (7).

In Figure 2, we plot, for  $n = 50$  (left) and  $n = 100$  (right) and  $p = .4$  (blue),  $p = .6$  (red), and  $p = .8$  (green) the value of  $m$  vs.  $\lambda_2$  for 5,000 randomly generated Erdős-Rényi graphs. The mean values obtained are indicated by circles. We use the greedy algorithm described in §3.1 (see Algorithm 1) with initial graph taken to be the path with  $n$  vertices,  $P_n$ , to compute nearly-optimal graphs with  $n$ -nodes and  $m$ -edges. The solid black line in Figure 2 represents the value of  $\lambda_2$  for these graphs. Finally, the dashed blue line in Figure 2 represents the upper bound on  $\lambda_2$  given in (7) (compare also to (13)).

We observe in Figure 2 that nearly-optimal graphs have values which are indeed close to the upper bound on the algebraic connectivity, indicating (i) the upper bound is nearly-tight

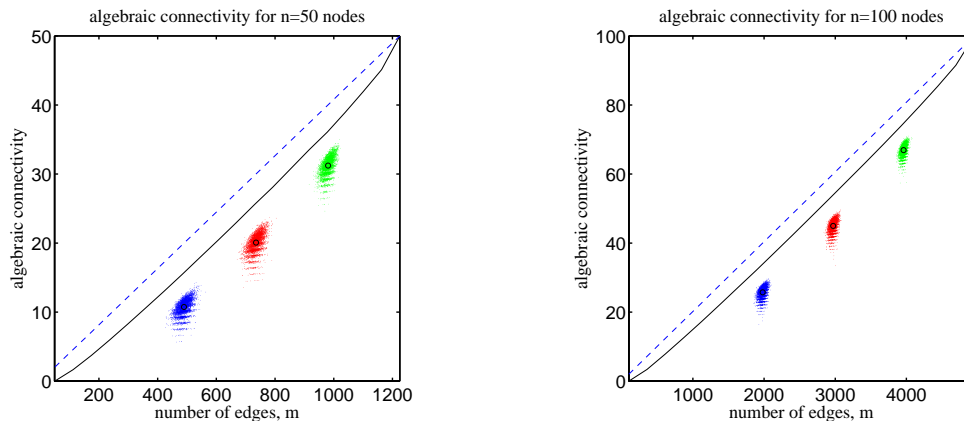


Figure 2: Algebraic connectivity,  $\lambda_2$  as a function of  $m$  for 50- and 100-node graphs. The dashed blue line represents the upper bound on  $\lambda_2$  given in (7). The solid black line represents the nearly-optimal value of  $\lambda_2$ . Finally, for  $p = .4$  (blue),  $.6$  (red), and  $.8$  (green) we give a scatter plot of  $(m, \lambda_2)$  for 5,000 randomly generated Erdős-Rényi graphs. The mean values obtained are indicated by circles. See §5.2.

and (ii) the greedy heuristic (Algorithm 1) produces graphs which are nearly-optimal. We also observe that the algebraic connectivity of nearly-optimal graphs is significantly better than the values for an average Erdős-Rényi random graph.

### 5.3 2011-12 NCAA Division I football schedule

In this section, we study the 2011-12 NCAA Division 1 football schedule downloaded from Massey Ratings<sup>2</sup>. As discussed in §1, the NCAA Division 1 Football League is divided into the Football Bowl Subdivision (FBS) and Football Championship Subdivision (FCS)<sup>3</sup>. The FBS is further decomposed into 12 conferences and the FCS into 15. Of the 246 teams in Division 1, 120 belong to FBS and 126 belong to FCS. Lafayette College is a member of FBS, however every opponent of Lafayette during the 2011-12 season was a member of the FCS. For our purposes, it is more convenient to reclassify Lafayette as a member of FCS and thus, in what follows, FBS has 119 teams and FCS has 127. There were  $m = 1430$  games amongst the Division 1 teams and  $m = 693$  games amongst the FBS teams.

#### 5.3.1 DATA VISUALIZATION VIA SPECTRAL CLUSTERING

We use the data visualization method described below to demonstrate that NCAA Division 1 teams primarily play against other teams within their own conference. We then show that this clustering of teams by conference results in the graph having poor algebraic connectivity.

We first use normalized spectral clustering to detect communities within the teams Shi and Malik (2000). This, in turn, relies on the  $k$ -means algorithm where  $k$  is the desired

2. <http://masseyratings.com/scores.php?t=11590&s=107811&all=1&mode=2&format=0>

3. These were formally known as Division 1-A and 1-AA respectively.

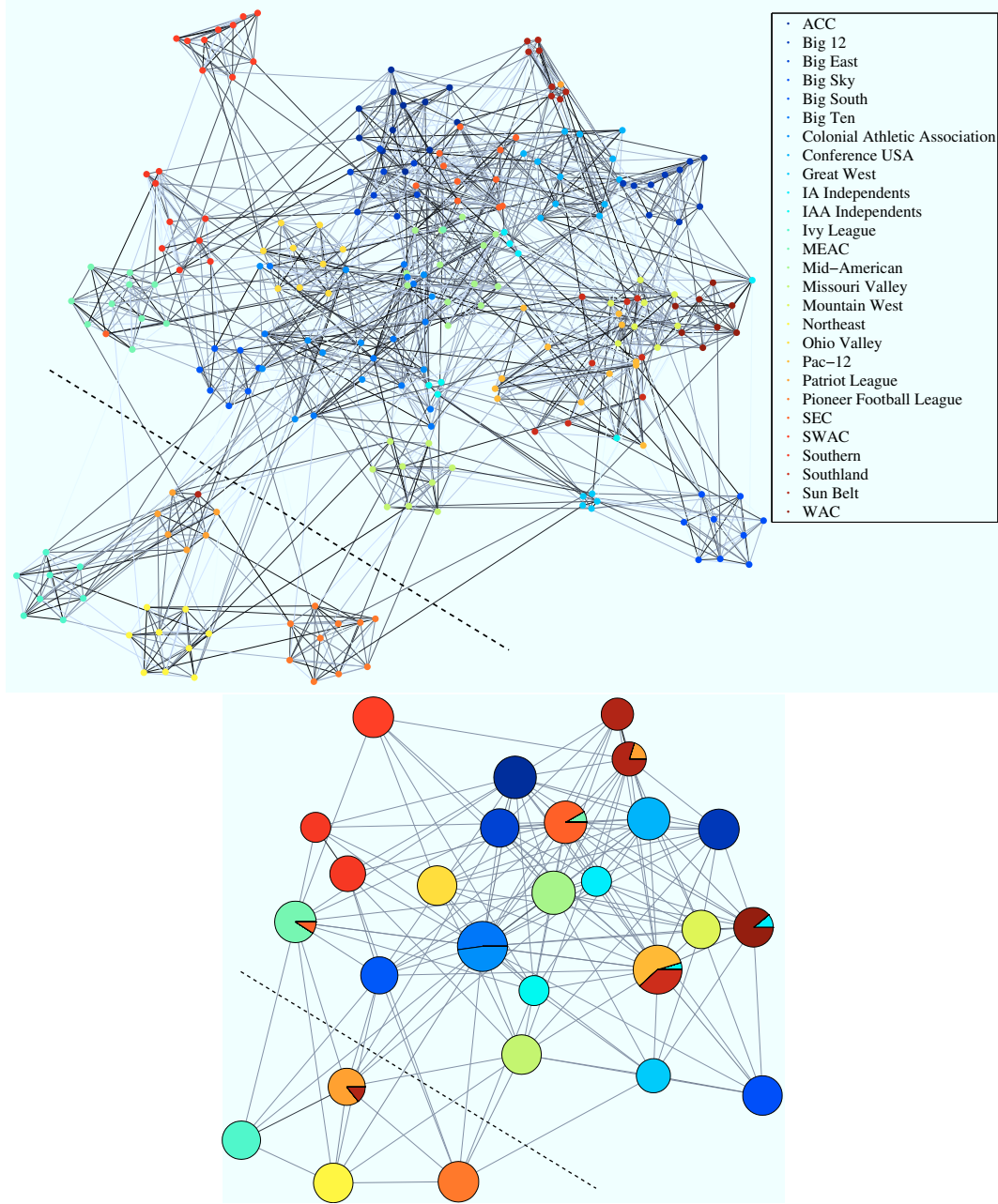


Figure 3: **2011-12 NCAA Division 1 (FBS and FCS) football schedule.** Graph representation of schedule via spectral clustering by games, *top*: vertices represent teams, edges represent games, coloring indicates conference membership. *bottom*: community detection of teams (represented using pie-graphs) reveals that teams primarily play within their own conference. The dashed lines indicate an edge cut which is discussed in the text. See §5.3.

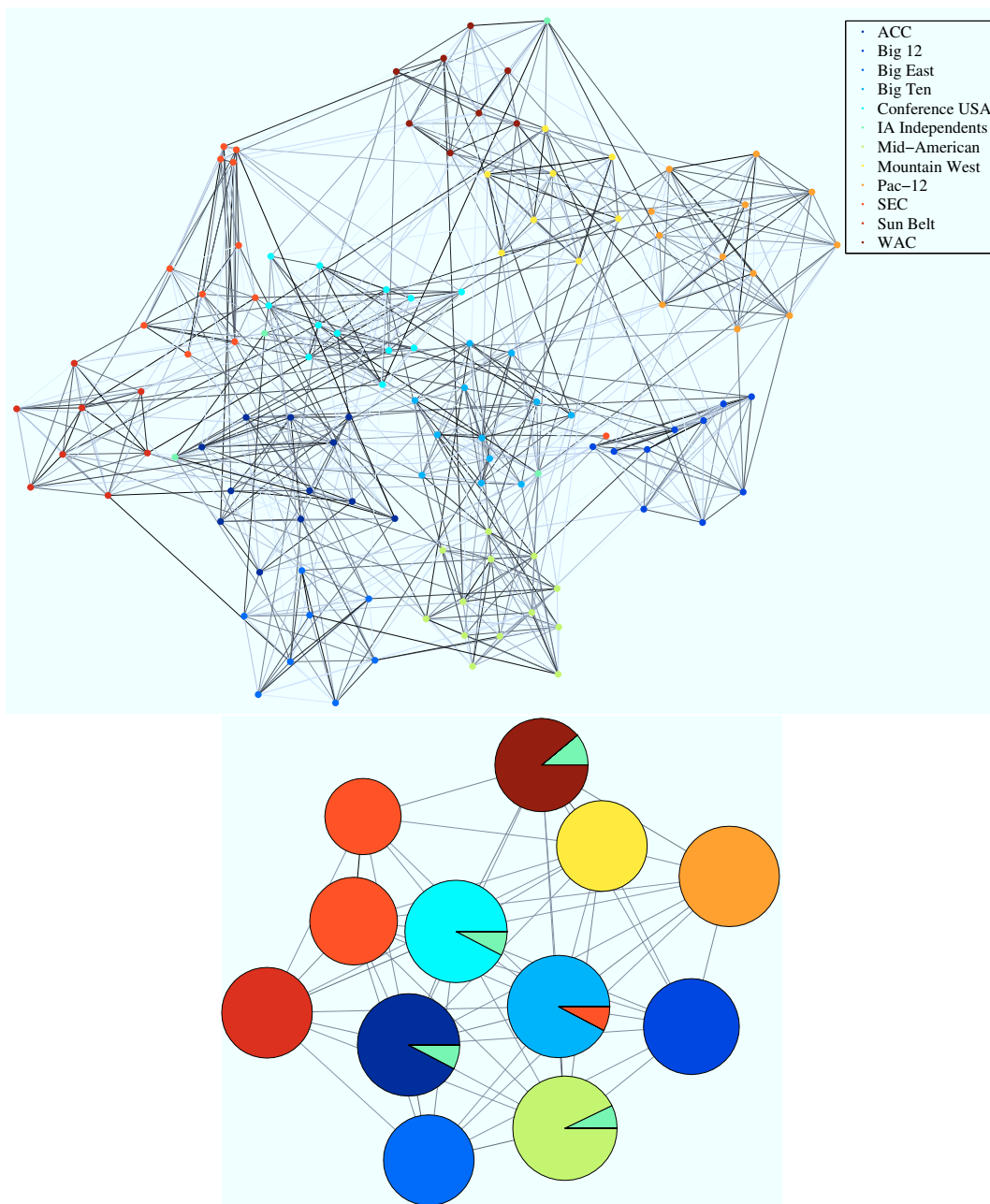


Figure 4: **2011-12 NCAA Division 1 (only FBS) football schedule.** Graph representation of schedule via spectral clustering by games, *top*: vertices represent teams, edges represent games, coloring indicates conference membership. *bottom*: community detection of teams (represented using pie-graphs) reveals that teams primarily play within their own conference. See §5.3.

number of communities (27 for Division 1 and 12 for Division 1 FBS). Then, using the Matlab toolbox described in [Traud et al. \(2009\)](#), the Fruchterman-Reingold algorithm finds an optimal placement of the communities and the Kamada-Kawai algorithm is used for the placement of nodes within each community. The mean within-cluster sum of point-to-centroid distances for the  $k$ -means clustering obtained for the Division 1 and Division 1 FBS data is 0.147 and 0.133 respectively.

In Figures 3 and 4, we plot the 2011-12 NCAA Division 1 and Division 1 FBS football schedules respectively. In 3(top) and 4(top), the vertices represent teams, the edges represent games, and each vertex (team) is colored by conference membership. In 3(bottom) and 4(bottom), the vertices represent the spectrally clustered communities and the edges represent the community interactions. We observe from Figures 3 and 4 that the teams primarily play within their own conference, which has implications which we discuss below.

We next compare the value of the algebraic connectivity for these schedules with schedules from Erdős-Rényi random graphs and proposed nearly-optimal schedules.

### 5.3.2 COMPARISON OF NCAA DIVISION 1, ERDŐS-RÉNYI RANDOM, AND NEARLY-OPTIMAL SCHEDULES

In the introduction, we noted that there are several common scalar measures of  $\text{Var}(\hat{\phi}_2)$ , three of which are given in (3). In this section, we compare these various measures for the NCAA Division 1, Erdős-Rényi random, and nearly-optimal schedules.

More concretely, let  $w$  be a given schedule (defining a graph on  $n$  vertices) and define the graph Laplacian:  $\Delta_w := B^t[\text{diag}(w)]B$ . Define the following three functions of  $w$ :

$$J_E(w) := \lambda_2(\Delta_w) \quad (14a)$$

$$J_A(w) := \left[ \frac{1}{n} \text{tr}(\Delta_w^\dagger) \right]^{-1} = \left[ \frac{1}{n} \sum_i \frac{1}{\lambda_i(\Delta_w)} \right]^{-1} \quad (14b)$$

$$J_D(w) := -\log[\det(\Delta_w^\dagger)]^{\frac{1}{n}} = \frac{1}{n} \sum_{i: \lambda_i \neq 0} \log[\lambda_i(\Delta_w)] \quad (14c)$$

Note that as defined, it is desirable to maximize  $J_E$ ,  $J_A$ , and  $J_D$  in (14) while it is desirable to minimize the quantities defined in (3).

For the Division 1 and Division 1 FBS schedules, we compute the various measures of the quality of schedule given in (14) and record them in Table 2. We also plot  $J_E(w)$  given in (14a) in Fig. 5 by a red diamond. We next discuss schedules for which we compare the Division 1 and Division 1 FBS schedules in Table 2 and Fig. 5.

The expected number of edges for a  $G(n, p)$  Erdős-Rényi random graph is  $pN$  where  $N := \binom{n}{2}$ . To compare to the football schedules, we take  $p = m/N$  and consider the family of random graphs,  $G(n, m/N)$ . For  $n = 119$  and  $m = 693$ , we choose  $p = m/N \approx 0.0987$  which is approximately 2.5 times the threshold for connectivity,  $p_c = \log(n)/n \approx 0.0402$ . For  $n = 246$  and  $m = 1430$ , we choose  $p = m/N \approx 0.0475$  which is approximately 2.1 times the threshold for connectivity,  $p_c = \log(n)/n \approx 0.0224$ . In Table 2, we tabulate the expected values of the three quantities given in (14) for  $G(n, m/N)$  graphs, obtained by averaging over a sample size of 1000. Similar to §5.2, in Fig. 5, we give a scatter plot of  $(m, \lambda_2)$  for  $G(n, m/N)$  graphs and indicate the mean values with a blue circle.



As in §5.2, we again use the greedy algorithm described in §3.1 (see Algorithm 1) to compute graphs with  $n$  nodes and  $m$  edges which nearly-maximize  $J_E = \lambda_2$ . We then evaluate all three quantities given in (14) for these graphs and tabulate these values in Table 2. The solid black line in Fig. 5 is the best value of  $J_E = \lambda_2$  obtained. Finally, the dashed blue line in Fig. 5 represents the upper bound on  $\lambda_2$  given in (7).

We observe in Fig. 5 and Table 2 that the schedules which nearly-maximize  $J_E(w) = \lambda_2$  have significantly larger values of  $J_E$  than the NCAA Division 1 and Division 1 FBS schedules. In fact, the NCAA schedules have worse values than schedules associated with Erdős-Rényi random graphs of the same size. Furthermore, we show in Table 2 that schedules which maximize  $J_E$  also have larger values of  $J_A$  and  $J_D$ . That is, the schedules which are good in the sense of E-optimality are also good schedules in the sense of D- and E-optimality as defined in (3).

The reason for the relatively poor value of  $J_E(w) = \lambda_2$  for the NCAA Division 1 and Division 1 FBS schedules can be understood from Figures 3 and 4, which are described in §5.3.1. Figures 3 and 4 reveal that teams primarily play within their own conference. This results in a small edge cut between a conference (or set of conferences) and its vertex complement, which, by (6), implies a small algebraic connectivity. For example, the edge cut indicated by the dashed line in Fig. 3 (entire NCAA Division 1 schedule) results in an upper bound on the algebraic connectivity of 1.297. The edge cut obtained by considering the set consisting of teams in the SWAC conference yields an upper bound equal to 1.043. Both of these bounds are already less than the expected value of  $\lambda_2$  for Erdős-Rényi random graphs of comparable size (compare with the top part of the first column in Table 2). To summarize, the NCAA primarily schedules games amongst teams within the same conferences and this reduces the informativeness of the rankings.

	$J_E(w)$ in (14a)	$J_A(w)$ in (14b)	$J_D(w)$ in (14c)
Div. 1 FBS and FCS	0.7015	8.780	2.363
Erdős-Rényi, $n = 246$	2.892	9.681	2.358
E-optimal design, $n = 246$	<b>6.630</b>	<b>10.71</b>	<b>2.403</b>
Div. 1 FBS	1.725	9.634	2.372
Erdős-Rényi, $n = 119$	3.497	9.911	2.361
E-optimal design, $n = 119$	<b>7.142</b>	<b>10.92</b>	<b>2.402</b>

Table 2: A comparison of the three objective functions defined in (14) for the Division 1 and Division 1 FBS schedules, Erdős-Rényi random schedules, and schedules which nearly-maximize  $J_E(w) = \lambda_2$ . Schedules which nearly-maximize  $J_E(w) = \lambda_2$  also have larger values of  $J_A$  and  $J_D$  than the comparison schedules. See §5.3

## 6. Discussion and future directions

We have applied methods from optimal experiment design to provide a new framework for designing more informative schedules, which reduce the variance in ranking. At the heart of this framework is an optimization problem (2) where the inner problem is to determine

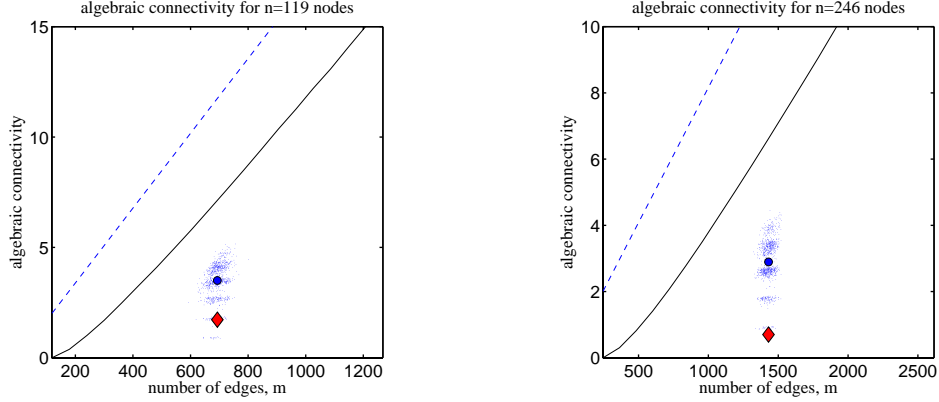


Figure 5: A comparison of  $J_E(w) = \lambda_2$  defined in (14a) for the Division 1 and Division 1 FBS schedules, Erdős-Rényi random schedules, and schedules which nearly-maximize  $\lambda_2$ . The red diamonds represents the 2011 NCAA Division 1 (right) and Division 1 FBS (left) football schedule. The solid black lines represent the nearly-optimal values of  $\lambda_2$  obtained for  $n = 119$  (left) and  $n = 246$  (right). The dashed blue lines represent the upper bound on  $\lambda_2$  given in (7). The blue dots represent a scatter plot of  $(m, \lambda_2)$  for 1,000 randomly generated Erdős-Rényi graphs,  $G(n, m/N)$ . The mean values are indicated by blue circles. See §5.3.

the unbiased ranking for a given schedule and the outer problem is to design the schedule which minimizes the variance of the ranking. We illustrate this method for the least-squares ranking estimate, demonstrating that in this case, the outer problem decouples from the inner problem and reduces to an eigenvalue optimization problem (11). For the E-optimal schedule, the eigenvalue optimization problem is to maximize the second eigenvalue of the graph Laplacian, referred to as the algebraic connectivity. Graph theoretic results then describe characteristics of graphs with large algebraic connectivity.

There are several applications in which improved scheduling can benefit ranking. In particular, we have demonstrated that graphs can be generated which represent schedules which are more informative than the 2011 NCAA Division 1 football schedule. More generally, the scheduling methods developed here could be implemented in any situation where there is a large number of items to be ranked and a relatively small number of pairwise comparisons available. This includes, for instance, a film festival where a large number of films must be reviewed in a short amount of time by a small number of reviewers [Xu et al. \(2011\)](#). Additionally, the scheduling method proposed here could be used to determine which pairwise comparison data to collect in modern internet and e-commerce applications, where the collection of data can have associated cost.

In the case of NCAA Division 1 football, we demonstrated in §5.3 and Table 2 that the nearly-optimal schedule in the sense of E-optimality is also a good schedule in the sense of D- and E-optimality; the choice of scalar function  $f: \mathbb{S}_+^n \rightarrow \mathbb{R}$  as defined in (2) does *not* strongly effect the optimal schedule (see Remark 1).

The schedule design methodology advocated in Eq. (11) is flexible in the following two senses: (i) The optimal schedules contain symmetry with respect to permutations in the seeding of the teams. This problem has been studied previously for tournaments; see the discussion in §2. (ii) The optimal schedule is *not* time dependent and thus the scheduling of future games does *not* depend on past game performances, *i.e.*, the schedule is completely known before the season begins and the games may be played in *any* order. These properties can be exploited in the further design of the schedule.

In this paper, we have focused on scheduling for improved rankings, neglecting several other influential factors, including traveling limitations and preferences of games between particular teams, *e.g.*, “rival teams”. There are two simple methods which may be employed to accommodate these additional factors. First, travel limitations or other financial aspects of gameplay can be incorporated by either adding a penalization term in (11) or by incorporating additional weights into the norm used to compute  $\lambda_2$  in (11). Schedules with games between particular teams may be obtained by explicitly adding these edges to the input graph of the greedy Algorithm 1 for computing nearly-optimal schedules.

We are interested in extending this work to nonlinear ranking methods, including robust estimators Osting et al. (2012), random walker methods Callaghan et al. (2007), Perron-Frobenius eigenvalue methods Keener (1993); Langville and Meyer (2012), and Elo methods Glickman (1995); Langville and Meyer (2012).

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